Homework 11 – STAT 543

On campus: Due Friday, April 20 by 5:00 pm (TA’s office);
you also may turn in the assignment in class on the same Friday
Distance students: Due Friday, April 27 by 5:00 pm (TA’s email)

1. Let $X_1, \ldots, X_n$ be iid exponential($\theta$) and let $\hat{\theta}_n$ denote the MLE based on $X_1, \ldots, X_n$.
   (a) Find an asymptotically pivotal quantity based on $\sqrt{n}(\hat{\theta}_n - \theta)$.
   (b) Find a variance stabilizing transformation (VST) for $\{ \hat{\theta}_n \}$ and use this to determine a large
       sample confidence interval for $\theta$ with approximate confidence coefficient $1 - \alpha$.
   (c) Suppose a random sample $X_1, \ldots, X_{100}$ of $n = 100$ observations yields $\bar{x}_n = 1.835464$.
       Use this information to obtain a large sample confidence interval for $\theta$ based on a likelihood
       ratio statistic, which has approximate confidence coefficient 90%. (You should be able to
       numerically determine the interval.) Using this data, compute also a confidence interval with
       approximate confidence coefficient 90% using the VST approach from part(b).

2. For $\theta > 0$, suppose that $X_1, \ldots, X_n$ are iid Uniform(0, $\theta$) . Consider the MLE of $\theta$, $\hat{\theta}_n = \max\{X_1, X_2, \ldots, X_n\}$.
   (a) Prove that, in distribution, $n(\theta - \hat{\theta}_n) \xrightarrow{d} \text{Exponential}(\theta)$ as $n \to \infty$ (i.e., converges in distribution
       to an exponential with mean $\theta$).
       (Hint: Evaluate $P[n(\theta - \hat{\theta}_n) > t]$ and remember that $\lim_{s \to \infty}(1 + \frac{a}{s})^s = \exp(a)$.)
   (b) Argue carefully that $\{ \theta > 0 : \theta \leq n\hat{\theta}_n/(n - \log(20)) \}$ can be used as a one-sided (upper)
       confidence interval for $\theta$ with approximate confidence coefficient 95% (that is, show that the
       interval will have an approximate coverage probability of 95% for each $\theta > 0$).

3. Suppose that $X_1, X_2, \ldots, X_n$ are a random sample of discrete random variables taking values in the
   range $\{1, 2, 3\}$ with probability mass function as
   \[
   f(x|p) = \begin{cases} 
   1 & x = 1 \\
   2p & x = 2 \\
   3p & x = 3
   \end{cases}, \quad p \in (0, 1/2).
   \]
   (a) Show the likelihood function for $p$ is given by
   \[L(p) = p^{S_n}(1 - 2p)^{n - S_n}, \quad \text{where} \quad S_n = \sum_{i=1}^{n} (X_i - 2)^2.\]
   (b) Find the maximum likelihood estimator $\hat{p}_n$ of $p$. (Please ignore the fact that $\hat{p}_n$
       technically does not exist when $S_n = 0$ or $S_n = n$.)
   (c) Show that, as $n \to \infty$, as the limiting distribution of $\sqrt{n}(\hat{p}_n - p)$ is normal $N\{0, 2^{-1}p(1 - 2p)\}$
       with mean 0 and variance ???.
   (d) Let $\sin^{-1}(x) : [0, 1] \to [0, \pi/2]$ denote the inverse of the sine function, which satisfies
       \[
       \sin(\sin^{-1}(x)) = x \quad \text{and} \quad \frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad x \in (0, 1).
       \]
       Find the limiting distribution of $\sqrt{n}\{h(\hat{p}_n) - h(p)\}$, where $h(x) = 2\sin^{-1}(\sqrt{2x})$.
   (e) Using the result above, give the lower limit $p_L$ in an approximate one-sided $(1 - \alpha)$ confidence